Vacuum Energy of $CP(1)$ Solitons

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Abstract

The vacuum energy of two $CP(1)$ solitons on a torus is computed numerically. A numerical technique for the zeta-function regularisation is proposed to remove the divergence of the vacuum energy. After performing the numerical regularisation, we observe the effect of the vacuum energy on the two-soliton configuration.

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I. INTRODUCTION

Topological solitons arise as solutions of classical field equations and corresdond to points where the classical energy attains degenerate local minima. The classical energy of a topological soliton typically depends on an integer which can be interpreted as the number of solitons, and in some cases it is independent of the parameters which correspond to the positions and the sizes of individual solitons. Therefore the quantum corrections can play a dominant role in the interactions between solitons.

Many techniques have been used for the numerical computation of the vacuum energies of single solitons [1–4]. These methods rely heavily on the spherical symmetry which is usually implicit in the single soliton solutions. In this paper, we propose a new technique which does not rely on any symmetry and can therefore be applied to multiple soliton solutions.

We will evaluate the vacuum energy for the $CP(1)$ model in $d = 1 + 2$ dimensions. Previous work on single solitons in the $CP(1)$ model has used an approach based on heat kernel coefficients and phase shifts [7]. It was shown that the solitons are unstable to collapse due to the quantum corrections. The $CP(1)$ sigma model also exhibits multi-soliton solutions. We focus on the charge-2 case and study their interaction due to quantum corrections.

The quantisation of $CP(1)$ solitons can be performed by following the standard perturbation techniques invented by Schwinger [5] and developed in the 1970s by several authors (for a review, see [6]). The regularisation will be performed by the zeta-function technique, which we convert into a form that can be evaluated numerically. We examine how the finite one-loop energy depends on parameters such as the separation of the two solitons and their width.

II. $CP(1)$ SIGMA MODEL IN $(2+1)$ DIMENSIONS

The $CP(1)$ sigma model consists of a single complex scalar field taking a value in the one dimensional complex projective plane $CP(1)$. The action of the $CP(1)$ sigma model is given by

$$S = \int d^3x \, \frac{|\partial_\mu u|^2}{(1+|u|^2)^2} \tag{1}$$

where μ is the spacetime index running over $0, 1, 2$ and $|u|^2 = u\bar{u}$. The complex projective plane being mapped to a sphere stereographically, the $CP(1)$ model is equivalent to the $O(3)$ model. Let u and $\{\phi^a\}$, $a = 1, 2, 3$ be coordinates of the $CP(1)$ and $O(3)$ respectively. The stereographic coordinates are given by

$$(u_1, u_2) = \left(\frac{\phi^1}{1-\phi^3}, \frac{\phi^2}{1-\phi^3} \right) \tag{2}$$

with $u = u_1 + iu_2$ and $(\phi^a)^2 = 1$. Expressing the action (1) in terms of ϕ^a, one can recover the $O(3)$ sigma model action

$$S = \frac{1}{4} \int d^3x \, (\partial_\mu \phi^a)^2 \, . \tag{3}$$

Topological soliton solutions for the $O(3)$, or equivalently $CP(1)$ model, were discovered by Belavin *et al.* [8]. To be precise, their solutions are instantons in two-dimensional Euclidean spacetime, but at a classical level, they are the same as solitons in three-dimensional Minkowski spacetime.

The static energy of the $O(3)$ model is given by

$$E = \frac{1}{4}\int d^2x\ (\partial_i \phi^a)^2 \tag{4}$$

where i takes the values $1, 2$. The finite energy condition requires a boundary condition

$$\partial_i \phi^a \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty\ . \tag{5}$$

Without loss of generality, we can define the asymptotic value of ϕ^a as

$$(\phi^a)^2 \to 1 \quad \text{as} \quad |\mathbf{x}| \to \infty \tag{6}$$

which satisfies (5). This boundary condition compactifies the space into a sphere. Since the field space is also a sphere, the homotopy group of the fields is $\pi_2(S^2) = Z$ and hence there are soliton solutions. The topological charge is given by

$$Q = \frac{1}{8\pi}\int d^2x\ \epsilon_{ij}\epsilon_{abc}\phi^a(\partial_i\phi^b)(\partial_j\phi^c)\ . \tag{7}$$

From the obvious identity

$$(\partial_i\phi^a - \epsilon_{ij}\epsilon_{abc}\phi^b\partial_j\phi^c)^2 \geq 0\ , \tag{8}$$

one can derive

$$(\partial_i\phi^a)^2 \geq \epsilon_{ij}\epsilon_{abc}\phi^a(\partial_i\phi^b)(\partial_j\phi^c)\ . \tag{9}$$

This implies

$$E \geq 2\pi|Q|\ . \tag{10}$$

The soliton solutions attain minimum energy in each topological sector and hence saturate the equality. Thus they satisfy the first order differential equation

$$\partial_i\phi^a - \epsilon_{ij}\epsilon_{abc}\phi^b\partial_j\phi^c = 0\ . \tag{11}$$

This equation turns out to be equivalent to the Cauchy-Riemann conditions in the $CP(1)$ version

$$\frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial y}, \qquad \frac{\partial u_2}{\partial x} = -\frac{\partial u_1}{\partial y}\ . \tag{12}$$

The general solution is given by the analytic function

$$u_0(z) = \frac{(z-b_1)\cdots(z-b_m)}{(z-a_1)\cdots(z-a_n)} \tag{13}$$

where $z = x + iy$, and $a_i, (i = 1, \cdots n)$ and $b_j, (j = 1, \cdots m)$ are the complex parameters characterising the position and size of solitons respectively. Note that the only singularities the field u can have are isolated poles, since they merely correspond to the north pole of a sphere in the target space of the $O(3)$ model.

The degree of the mapping is equal to the topological charge of the solution (13), which is also equal to the number of solutions when expressing z in terms of u_0,

$$Q = \max(m, n) \; . \tag{14}$$

$\bar{u}$ corresponds to the anti-soliton solution and gives an opposite charge of u.

III. CHARGE-2 $CP(1)$ SOLITONS ON THE TORUS

For the sake of numerical work, we consider the $CP(1)$ model on the torus [9]. This requires a slight modification of the previous soliton solutions. We shall assume that the field $u(z)$ on the torus satisfies the periodic boundary condition

$$u(z + 1 + i) = u(z) \tag{15}$$

The fact that the only singularities of $u_0(z)$ are poles means $u_0(z)$ is a meromorphic function. Then $u_0(z)$ can be represented as an elliptic function.

In the charge-2 case, the solution is represented by the Weierstrass function (see appendix A). For simplicity, we shall assume that the two solitons are symmetric in their size and location. Then there are only two complex parameters required which are the degrees of freedom of the size and separation. Thus we may write

$$u_0(z) = \frac{1}{\alpha(\wp(z) + \rho)} \tag{16}$$

with complex parameters α and ρ.

Let the separation of the solitons (located at the poles) and the width be 2ϵ and w respectively and restrict to real parameters. For (16), one obtains

$$\rho = -\wp(i\epsilon), \qquad \alpha = \frac{2}{w\wp'(i\epsilon)} \; . \tag{17}$$

In terms of ϵ and w, the solution (16) becomes

$$u_0(z) = \frac{w\wp'(i\epsilon)}{2(\wp(z) - \wp(i\epsilon))} \; . \tag{18}$$

The energy density for this solution with $\epsilon = 0.3$ and $w = 0.5$ is plotted in fig.1.

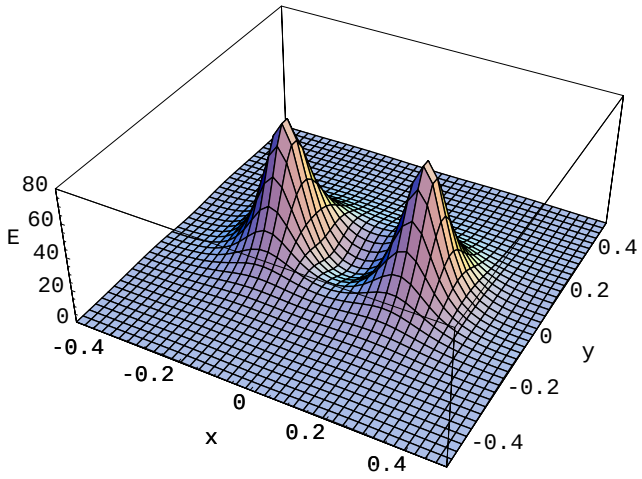

FIG. 1. Energy density of charge 2-soliton for $\epsilon = 0.3$ and $w = 0.5$.

IV. VACUUM ENERGY OF CHARGE-2 $CP(1)$ SOLITONS

The static energy functional for the $CP(1)$ model on the torus is given by

$$E[u] = \int_{T_2} d^2x \, \frac{|\partial_i u|^2}{(1+|u|^2)^2} \, . \tag{19}$$

The analytic expansion at u_0 is

$$E[u] = E_0 + \int_{T_2} d^2x \; G\,\bar{\xi}\Delta_f \xi + \cdots \tag{20}$$

where $\xi(z,\bar{z}) = u(z,\bar{z}) - u_0(z)$. The 'metric' G and the fluctuation operator Δ_f are defined by

$$G = \frac{1}{(1+|u_0|^2)^2} \tag{21}$$

and

$$\Delta_f \xi \equiv \left[-\nabla^2 + \frac{4\bar{u_0}(\partial_\mu u_0)}{1+|u_0|^2}\partial^\mu \right] \xi \tag{22}$$

respectively. The computation of the vacuum energy depends on solving the eigenvalue problem

$$\Delta_f \chi_n = \lambda_n \chi_n \, . \tag{23}$$

Zero eigenvalues correspond to changes in the parameters, or moduli, of the soliton solution. The quantisation of these parameters is treated separately. Expanding ξ in terms of the remaining eigenfunctions gives an infinite sequence of (degeneracy two) oscillators with

vacuum energy $\lambda_n^{1/2}$. The total vacuum energy can be regularised using the zeta-function scheme [16,15],

$$E = E_0 + \zeta\left(-\tfrac{1}{2}\right) \tag{24}$$

where the sum over non-zero eigenvalues

$$\zeta(s) = \sum_{n=1}^{\infty} \lambda_n^{-s} \ . \tag{25}$$

gives the generalised Riemann zeta function for $s > 1$. The value at $s = -1/2$ is uniquely determined by analytic continuation.

V. NUMERICAL REGULARISATION

We will now show how the zeta function regularisation scheme, which depends on analytic continuation, can be converted into a numerical subtraction scheme. Suppose that on average the eigenvalues approach the form

$$\widehat{\lambda}_n = an + b \ . \tag{26}$$

Using (26), we can define the corresponding heat kernel

$$\widehat{K}(t) = \sum_{n=1}^{\infty} e^{-\hat{\lambda}_n t} \ . \tag{27}$$

The behiour of the heat kernel as $t \to 0$ gives information about the eigenvalues at large values of n. We have

$$\widehat{K}(t) = \frac{e^{-bt}}{1 - e^{-at}} = \frac{1}{at} \sum_{n=0}^{\infty} b_n \left(\frac{b}{a}\right) \frac{(at)^n}{n!} (-1)^n \tag{28}$$

where $b_n(x)$ are the Bernoulli polynomials and the first three terms are given by

$$b_0(x) = 1 \ \ , \ \ b_1(x) = x - \frac{1}{2} \ \ , \ \ b_2(x) = x^2 - x + \frac{1}{6} \ . \tag{29}$$

For small t, the leading terms are

$$\widehat{K}(t) = \frac{1}{at} - \left(\frac{b}{a} - \frac{1}{2}\right) + O(t) \ . \tag{30}$$

Comparing the order of t in (B4) and (30), one can deduce

$$a = \frac{1}{B_0}, \qquad b = \frac{1}{B_0}\left(-B_1 + \frac{1}{2}\right) \ . \tag{31}$$

where B_0 and B_1 are the heat kernel coefficients for the operator (18) calculated in appendix B, $B_0 = 1/(4\pi)$ and $B_1 = 2$. Thus, $a = 4\pi$ and $b = -6\pi$, with the result that

$$\widehat{\lambda}_n = 4\pi n - 6\pi \ . \tag{32}$$

We assign a zeta function to this eigenvalue

$$\widehat{\zeta}(s) = \sum_{n=0}^{\infty} \widehat{\lambda}_n^{-s} = \left(\frac{1}{4\pi}\right)^s \zeta_H(s, \frac{1}{2}) \tag{33}$$

where $\zeta_H(s, \frac{1}{2})$ is the generalised zeta function defined by

$$\zeta_H(s, a) \equiv \sum_{n=0}^{\infty} (n + a)^{-s} \ . \tag{34}$$

Hurwitz has given a proof of the following formula

$$\zeta_H(s, a) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}} \left[\sin\left(\frac{s\pi}{2}\right) \sum_{n=1}^{\infty} n^{s-1} \cos(2n\pi a) + \cos\left(\frac{s\pi}{2}\right) \sum_{n=1}^{\infty} n^{s-1} \sin(2n\pi a) \right] \ .$$

Using this expression, one obtains

$$\widehat{\zeta}(s) = \left(\frac{1}{4\pi}\right)^s \frac{2\Gamma(1-s)}{(2\pi)^{1-s}} \sum_{n=1}^{\infty} n^{s-1} (-1)^n \sin\left(\frac{s\pi}{2}\right) \ . \tag{35}$$

Setting $s = -1/2$ gives

$$\widehat{\zeta}\left(-\frac{1}{2}\right) = \frac{1}{2\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{3/2}} \ . \tag{36}$$

We can obtain a finite sum by subtracting the divergent terms,

$$\zeta_{\rm reg}(s) = \lim_{N \to \infty} \sum_{n=1}^{N} \left(\lambda_n^{-s} - \widehat{\lambda}_n^{-s} \right) \tag{37}$$

where all the zero modes are removed from the eigenvalues. Then one can write the true zeta function as

$$\zeta(s) = \zeta_{\rm reg}(s) + \hat{\zeta}(s) \ . \tag{38}$$

Analytic continuation now implies

$$\zeta\left(-\frac{1}{2}\right) = \zeta_{\rm reg}\left(-\frac{1}{2}\right) + \hat{\zeta}\left(-\frac{1}{2}\right) \ . \tag{39}$$

The above expression allows us to evaluate the one-loop energy of the solitons numerically, assuming that (37) can be evaluated numerically. In practice, there are problems with the limit $N \to \infty$ at $s = -1/2$ because the residuals $\lambda_n - \widehat{\lambda}_n$ do not vanish as $n \to \infty$. However, we shall see in later sections how the difference does vanish 'on average', allowing a result to be obtained.

VI. NUMERICAL COMPUTATION

We use the Jacobi theta functions as a representation of the Weierstrass $\wp$ function (see appendix A)

$$\wp(z) = \pi^2 \left[\frac{\theta_2(0)\theta_4(0)\theta_3(\pi z)}{\theta_1(\pi z)} \right]^2 . \tag{40}$$

Thus the charge 2-soliton solution (18) can be written as

$$u_0(z) = i\pi w \frac{\theta_3^2(0)\theta_2(i\pi\epsilon)\theta_4(i\pi\epsilon)\theta_1^2(\pi z)}{\theta_1^2(i\pi\epsilon)\theta_3^2(\pi z) - \theta_3^2(i\pi\epsilon)\theta_1^2(\pi z)} . \tag{41}$$

In the following numerical computation, we shall use (41) as a soliton solution.

We shall evaluate the approximate eigenvalues ω in (23) by the Rayleigh-Ritz variational method. This method introduces two principles. Firstly, the eigenfunctions are stationary configurations of the functional

$$E[\xi] = \frac{\int_{T_2} d^2x \; G\bar{\xi}\Delta_f\xi}{\int_{T_2} d^2x \; G|\xi|^2} . \tag{42}$$

Thus the eigenfunctions satisfy

$$\frac{\delta E[\xi]}{\delta \xi} = 0 . \tag{43}$$

And secondly, trial functions which give stationary values of E provide upper bounds for the eigenvalues (Hylleraas-Undheim theorem).

The trial functions can be constructed as follows. We take N linearly independent functions ϕ_n parametrised by N variational parameters c_n and construct the trial function as their superposition

$$\xi = \sum_{n=1}^{N} c_n \phi_n . \tag{44}$$

Inserting into (42) one obtains the functional E as a function of the N variational parameters

$$E[c_1, c_2, \cdots c_N] = \frac{\sum_{m,n=1}^{N} \bar{c}_m c_n A_{mn}}{\sum_{m,n=1}^{N} \bar{c}_m c_n B_{mn}} \tag{45}$$

where we defined

$$A_{mn} = \int_{T_2} d^2x \; G\bar{\phi}_m \Delta_f \phi_n \tag{46}$$

$$B_{mn} = \int_{T_2} d^2x \; G\bar{\phi}_m \phi_n . \tag{47}$$

Then, from the minimum principle, the upper bound of the nth eigenvalue is given by the stationary point of $E[c_1, c_2, \cdots c_N]$ with respect to c_n, i.e. E satisfies

$$\frac{\partial E[c_1, c_2, \cdots c_N]}{\partial c_n} = 0 \ . \tag{48}$$

Thus we obtain N linear homogeneous equations for each upper bound of E_n, $\quad (n = 1, \cdots N)$

$$\sum_{m=1}^{N} \bar{c}_m (A_{mn} - E B_{mn}) = 0 \tag{49}$$

and the problem is reduced to the nth-degree secular equation of the $N \times N$ matrix

$$\det(\mathbf{A} - E\mathbf{B}) = 0 \ . \tag{50}$$

Increasing in the number of basis states gives a lower upper bound of the exact eigenvalue in each mode.

We take the basis of trial functions $\phi_{\vec{k}}$ to be plane waves

$$\phi_{\vec{k}} = \exp\left\{\frac{i}{2}(\bar{k}z + k\bar{z})\right\} \tag{51}$$

where $k = \frac{\pi}{L}(n+im) = 2\pi(n+im)$ and $\vec{k} = (n, m)$, $\quad n, m = -N, -N+1, \cdots, 0, \cdots, N-1, N$. n and m are the mode numbers in the x and y directions respectively. With this basis, the functional (45) becomes

$$E[\bar{c}_{\vec{k}'}, c_{\vec{k}}] = \frac{\bar{c}_{\vec{k}'} c_{\vec{k}} A_{\bar{k}'k}}{B_{\bar{k}'k}} \tag{52}$$

where

$$A_{\bar{k}'k} = \int d\bar{z}dz \frac{1}{(1+|u_0|^2)^2} \bar{k}'k \exp\left[\frac{i}{2}\{(\bar{k}' - \bar{k})z + (k'-k)\bar{z}\}\right] \tag{53}$$

$$B_{\bar{k}'k} = \int d\bar{z}dz \frac{1}{(1+|u_0|^2)^2} \exp\left[\frac{i}{2}\{(\bar{k}' - \bar{k})z + (k'-k)\bar{z}\}\right] \ . \tag{54}$$

Therefore the upper bound of the spectrum can be computed from the secular equation of an $(2N+1) \times (2N+1)$ matrix

$$\det(A_{\bar{k}'k} - E B_{\bar{k}'k}) = 0 \ . \tag{55}$$

The spectrum will be more accurate as N increases and in the limit $N \to \infty$, it is exact.

To solve (55) numerically, we simply used the LAPACK (Linear Algebra Package) which provides routines for solving systems of simultaneous linear equations, least-square solutions of linear systems of equations, eigenvalue problems, and singular value problems. The file actually used is *chegv.f* for computing all eigenvalues of an Hermitian-definite generalised eigenproblem.

VII. NUMERICAL RESULTS

We have plotted the first 200 eigenvalues in fig.(2) as a function of the mode number $n = (\bar{k}, k)$ for a typical background solution. The agreement with the asymptotic formula (32) is quite striking. Fig.(3) is a plot of the regularised zeta function as a function of the number of modes N up to which we took the sum of the eigenvalues. The fluctuations in the data reflect the fact that the eigenvalues deviate from the asymtotic formula. Although we can easily improve the accuracy of the individual eigenvalues, these fluctuations have a significant influence in preventing the convergence of the zeta-function in all of the cases which we have examined. This problem has been seen before in numerical calculations of zeta-functions [10]. Averaging over N has been proposed as a solution to the problem and we use this in our calculation.

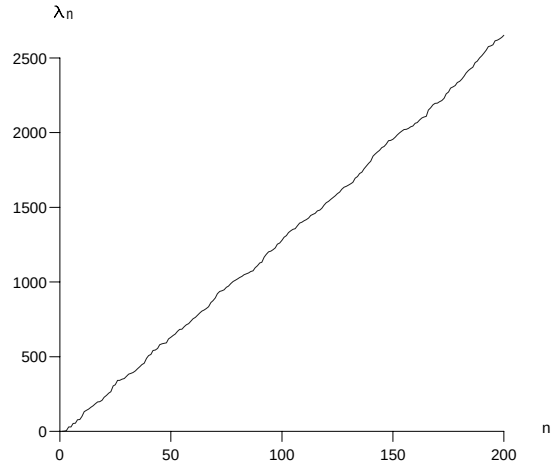

FIG. 2. nth-eigenvalue λ_n as a function of mode number n.

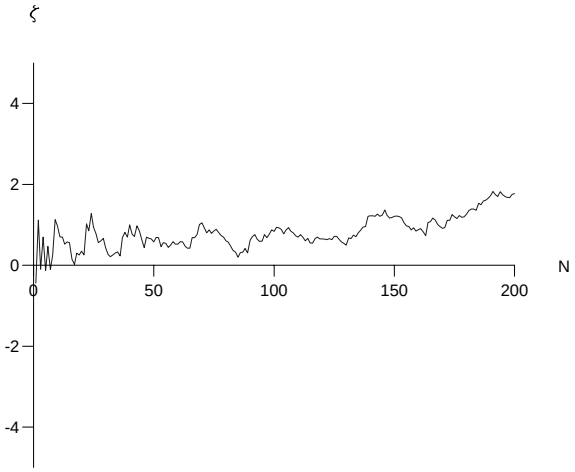

FIG. 3. Regulated zeta function as a function of the maximum mode number N.

Fig.(4) is a contour plot of the average value of the zeta function on the parameter space of ϵ and w. The data shows that the more spiky and closer two solitons are, the lower the vacuum energy is. Thus the classically stable two solitons become unstable to collapse and merger when the one-loop quantum correction is taken into consideration.

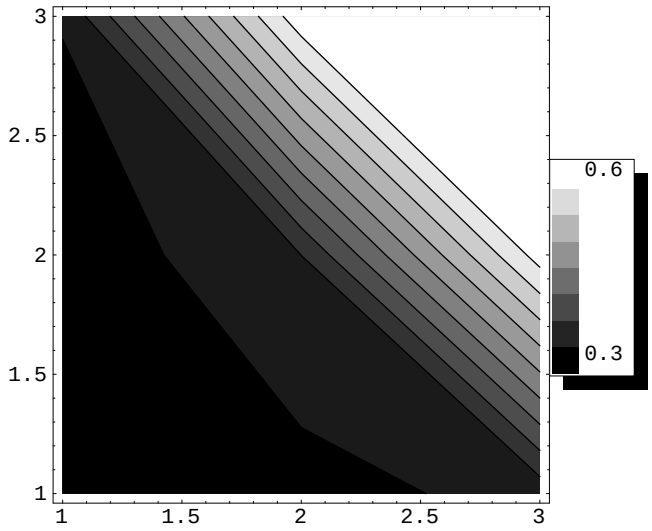

FIG. 4. One loop energy of two separated solitons as a function of ϵ and w.

As a special case of a charge-2 soliton, we also examined the one-loop energy for the solution

$$u_0(z) = \frac{w}{\wp(z)} \ . \tag{56}$$

This solution exhibits a ring-shape energy configuration as is shown in fig.(5). The numerical one-loop energy as a function of w is given in fig.(6). The plots denote the numerical values and the line is its interpolated function which is found to be

$$0.269\sqrt{w} \ . \tag{57}$$

The rotational symmetry implies that this is also a case that can be analysed using the phase shift technique [7]. Preliminary results suggest an identical functional form for the vacuum energy, but the phase shifts give a different overall scale [17].

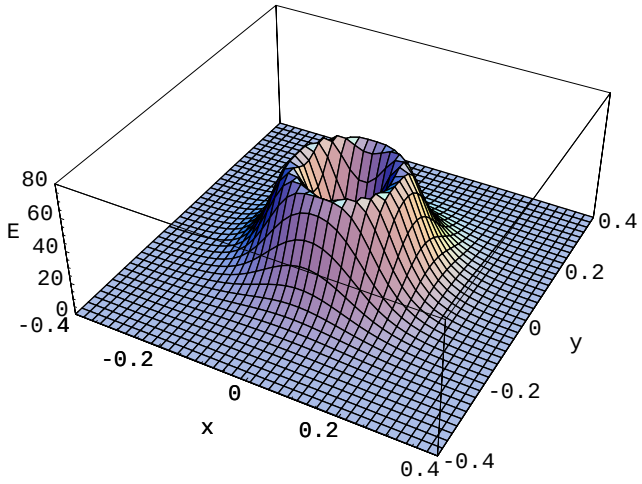

FIG. 5. Energy density of a ring for $w = 1.0$.

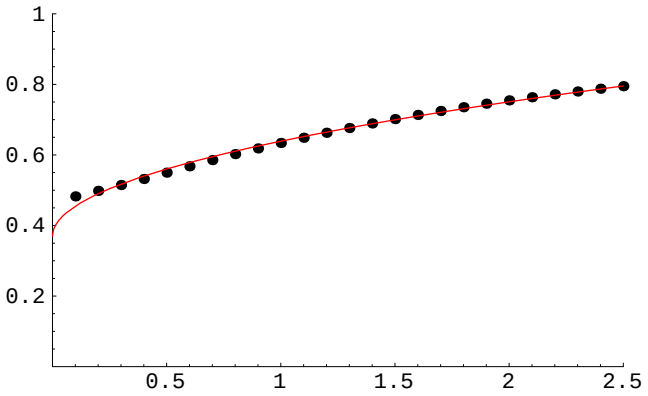

FIG. 6. One loop energy of a ring as a function of w.

VIII. CONCLUSION

In this paper, we have presented a numerical method for the computation of finite one-loop energies for solitons and studied its effect on the interaction of two $CP(1)$ solitons. The method does not require any symmetry to be present in the problem, but it is expensive in computer time and relies on the averaging out of residual terms in the eigenvalue spectrum. Further analysis of these eigenvalue residuals, which should be related to fundamental periods of the torus, would be useful and might improve the technique.

The results are very satisfactory and show that initially separated solitons are unstable to collapse and merger. Furthermore, the results are in qualitative agreement with the fast and reliable phase shift method [7] for a ring configuration, which has circular symmetry.

Interestingly, the vacuum energy is not the only effect on the dynamics of two interacting solitons. There are additional forces arising from the reduction of the quantum field theory to the $CP(1)$ moduli space [18]. In this example, an additional term is given by the Ricci tensor of the moduli space, and gives a repulsive force between the two solitons which is larger than the effect of the vacuum energy at very small separations [17].

In principle, our methods are applicable to other solitons such as skyrmions, although moving from two to three dimensions would hughely increase the computer time. The one-loop correction to the calculation of the skyrmion mass is significant. For physical values of the pion decay constant, the classical skyrmion has a mass about 50% larger than a nucleon mass. However, according to Moussallam and Kalafatis [11], including one loop energies within the framework of chiral perturbation theory, the nucleon mass can be correctly predicted to within 20%. Presumably, quantum effects within chiral perturbation theory would also have an effect on the force between two nucleons which could be calculated by the method presented here.

APPENDIX A: WEIERSTRASS ELLIPTIC FUNCTIONS

An elliptic function f is a function such that it is doubly periodic with two primitive periods $2\omega_1, 2\omega_2$ whose ratio is not real, i.e.

$$f(z + 2m\omega_1 + 2n\omega_2) = f(z) \tag{A1}$$

where m, n are integers and

$$\mathrm{Im}\left(\frac{\omega_1}{\omega_2}\right) \neq 0 \; . \tag{A2}$$

Thus for an elliptic function $f(z)$, the z-plane can be partitioned into parallelograms with vertices $z_0 + 2m\omega_1 + 2n\omega_2$. The only singularities of $f(z)$ allowed in a period parallelogram are poles.

The weierstrass function $\wp$ is an even elliptic function of order 2 with one double pole at $z = 2m\omega_1 + 2n\omega_2$ and defined by

$$\wp(z) = \wp(-z) = \frac{1}{z^2} + \sum_{m,n}\left[\frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2}\right] \tag{A3}$$

where m, n takes all integers except for $m = n = 0$. The series (A3) converges everywhere except at the poles.

The function $\wp(z)$ satisfies the differential equation

$$\left[\frac{d}{dz}\wp(z)\right]^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \equiv 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3) \tag{A4}$$

where g_2 and g_3 are invariants and determine the periods ω_1 and ω_2 as

$$g_2 = 60\sum_{m,n}\frac{1}{(m\omega_1 + n\omega_2)^4}, \quad g_3 = 140\sum_{m,n}\frac{1}{(m\omega_1 + n\omega_2)^6} \; . \tag{A5}$$

e_1, e_2 and e_3 are then given by

$$e_1 = \wp(\omega_1), \;\; e_2 = \wp(\omega_2), \;\; e_3 = \wp(\omega_2) \; . \tag{A6}$$

with

$$e_1 + e_2 + e_3 = 0, \quad e_1e_2 + e_2e_3 + e_3e_1 = -\frac{g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4} \ . \tag{A7}$$

The Weierstrass function can be represented in terms of Jacobi theta functions as

$$\wp(z) = e_j + \frac{\pi^2}{4\omega_1^2}\left[\frac{\theta_1'(0)\theta_{j+1}(v)}{\theta_{j+1}(0)\theta_1(v)}\right]^2 \qquad j = 1, 2, 3 \tag{A8}$$

where

$$v \equiv \frac{\pi z}{2\omega_1} \tag{A9}$$

and the theta functions are defined by

$$\begin{aligned}
\theta_1(z,q) \equiv \theta_1(z) &= 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin\{(2n+1)z\} \\
\theta_2(z,q) \equiv \theta_2(z) &= 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} q^{n(n+1)} \cos\{(2n+1)z\} \\
\theta_3(z,q) \equiv \theta_3(z) &= 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos(2nz) \\
\theta_4(z,q) \equiv \theta_4(z) &= 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2nz) \ ,
\end{aligned}$$

with $q = e^{i\omega_2/\omega_1}$. e_1, e_2 and e_3 can also be expressed by the theta functions

$$\sqrt{e_1 - e_2} = \frac{\pi}{2\omega_1}\theta_4^2(0), \quad \sqrt{e_2 - e_3} = \frac{\pi}{2\omega_1}\theta_2^2(0), \quad \sqrt{e_1 - e_3} = \frac{\pi}{2\omega_1}\theta_3^2(0) \ . \tag{A10}$$

We adopt the theta-function representation for the weierstrass function since the theta functions converge rapidly for n and the series are periodic. For definiteness, we take the Lemniscatic case

$$\omega_1 = -i\omega_2 = \frac{1}{2} \quad \text{and} \quad e_2 = 0 \ . \tag{A11}$$

Then from (A10), we can determine numerical values of e_1 and e_3 as

$$e_1 = -e_3 = \pi^2\theta_4^4(0) = \frac{1}{8\pi}\Gamma\left(\frac{1}{4}\right)^4 \sim 6.875 \ . \tag{A12}$$

Choosing $j = 2$ in (A8), one obtains

$$\wp(z) = \pi^2\left[\frac{\theta_2(0)\theta_4(0)\theta_3(\pi z)}{\theta_1(\pi z)}\right]^2 \ . \tag{A13}$$

APPENDIX B: HEAT KERNEL COEFFICIENTS

The small time expansion of a heat-kernel provides us a great deal of information about operators and their eigenvalues. This expansion was substantially developed in the 1960s, and more of the mathematical details can be found in the original work and reviews for example [12,14].

Consider the eigenvalue problem on a manifold M

$$\Delta\phi = \lambda\phi \tag{B1}$$

The field ϕ can have both spacetime, spinor and internal group indices, while Δ is the elliptic operator. The (integrated) heat-kernel of the operator Δ is defined by

$$K(t) = \sum_n e^{-\lambda_n t} \tag{B2}$$

We consider operators of the form

$$-D^2 + X \tag{B3}$$

with a gauge covariant derivative $D = \nabla + A$ acting on fields which are associated with a representation of some given gauge group. For these operators, Gilkey has shown that the function $K(t)$ has an asymptotic expansion

$$K(t) \sim t^{-d/2} \sum_{n=0}^{\infty} B_n t^n \tag{B4}$$

in d-dimensional space. In general, the B_n coefficients depend on the operator, the geometry of the manifold and the boundary conditions satisfied by the fields ϕ.

The coefficients B_0 and B_1 for the $CP(1)$ model can be obtained by applying the general expressions. First, we rewrite the fluctuation operator obtained in 22 in a covariant form

$$-\partial^\mu\partial_\mu + \frac{4\bar{u}_0(\partial_\mu u_0)}{1+|u_0|^2}\partial^\mu = -2\left(D_z D_{\bar{z}} + D_{\bar{z}} D_z\right) - \frac{4|\partial_z u_0|^2}{(1+|u_0|^2)^2} \tag{B5}$$

where D_z is a covariant derivative defined by

$$D_z u = \partial_z u - \frac{2\bar{u}_0(\partial_z u_0)}{1+|u_0|^2} u \ . \tag{B6}$$

Then the heat kernel coefficients for this operator are

$$B_0 = \frac{1}{4\pi}\int d^2x \, \mathrm{tr}(1) \tag{B7}$$

$$B_1 = \frac{1}{4\pi}\int d^2x \, (-X) = \frac{1}{\pi}\int d^2x \, \frac{|\partial_z u_0|^2}{(1+|u_0|^2)^2} \tag{B8}$$

Clearly, for a torus of unit area, $B_0 = 1/(4\pi)$. For a solution of the type

$$u_0(z) = \frac{\gamma}{z^2+\epsilon} \ , \tag{B9}$$

one can show directly that

$$B_1 = 2 \ . \tag{B10}$$

The same result can be seen to hold for the two-soliton solution on the torus, because the integral (B8) is the index of the mapping given by u.

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